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Group Theoretical Quantization of Phase and Modulus Related to Interferences

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Abstract

Following a recent group theoretical quantization of the symplectic space $\mathcal{S} = \{(\varphi \in \mathbb{R} \bmod 2\pi, p > 0)\}$ in terms of irreducible unitary representations of the group $SO^{\uparrow}(1,2)$ the present paper proposes an application of those results to the old problem of quantizing modulus and phase in interference phenomena: The self-adjoint Lie algebra generators K_1, K_2 and K_3 of that group correspond to the classical observables $p\cos\varphi, -p\sin\varphi$ and $p > 0$ the Poisson brackets of which obey that Lie algebra, too. For the irreducible unitary representations of the positive series the modulus operator K_3 has the positive discrete spectrum $\{n+k, n = 0,1,2,\ldots; k > 0\}$. Self-adjoint operators $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ can then be defined as $(K_3^{-1}K_1 + K_1K_3^{-1})/2$ and $-(K_3^{-1}K_2 + K_2K_3^{-1})/2$ which have the theoretically desired properties for $k \geq 0.32$. Some matrix elements with respect to number eigenstates and with respect to coherent states are calculated. One conclusion is that group theoretical quantization may be tested by quantum optical experiments.

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1 Introduction

In a recent paper [1] the symplectic manifold

$$\mathcal{S} = \{(\varphi \in \mathbb{R} \bmod 2\pi, p\ >\ 0)\} \tag{1}$$

(associated with the symplectic form $d\varphi \wedge dp$) was quantized group theoretically by means of the group $SO^{\uparrow}(1,2)$ (identity component of the proper Lorentz group in 2+1 space-time dimensions). The purpose was the quantization of Schwarzschild black holes [2]. In the meantime I realized that that quantization also sheds new light on the old unsolved problem how to represent phase and modulus as self-adjoint operators in a Hilbert space associated with the corresponding physical system (see the Reviews [3]).

The crucial point is that the manifold (1) has the nontrivial topology $S^1 \times \mathbb{R}^+$, $\mathbb{R}^+$: real numbers > 0. Such a manifold cannot be quantized in the usual naive way used for a phase space with the trivial topology $\mathbb{R}^2$ by converting the classical canonical pair (q,p) of phase space variables into operators and their Poisson bracket into a commutator. Here the group theoretical quantization scheme developed in the eighties of the last century as a generalization of the conventional one (see the reviews [4]) helps: The group $SO^{\uparrow}(1,2)$ acts symplectically, transitively, effectively and (globally) Hamilton-like on the manifold (1) and, therefore, its irreducible representations (or those of its covering groups) can provide the basic self-adjoint quantum observables and their Hilbert space of states (see Ref. [1] for more details): In the course of the group theoretical quantization one finds that the three basic classical observables

$$P_3 = p > 0, \quad P_1 = p\,\cos\varphi, \quad P_2 = -p\,\sin\varphi \tag{2}$$

correspond to the three self-adjoint Lie algebra generators $K_3\,, K_1\,,$ and K_2 of a positive discrete series irreducible unitary representation of the group $SO^{\uparrow}(1,2)$ or one of its infinitely many covering groups, e.g. the double covering $SU(1,1)$ which is isomorphic to the groups $SL(2,\mathbb{R})$ and $Sp(1,\mathbb{R})$. The generators K_i obey the commutation relations

$$[K_3, K_1] = iK_2, \quad [K_3, K_2] = -iK_1, \quad [K_1, K_2] = -iK_3 \quad . \tag{3}$$

Here K_3 is the generator of the compact sub-group $SO(2)$.

The corresponding Poisson brackets for the classical observables (2),

$$\{P_3, P_1\} = -P_2, \quad \{P_3, P_2\} = P_1, \quad \{P_1, P_2\} = P_3 \ , \tag{4}$$

form the real Lie algebra of $SO^{\uparrow}(1,2)$ ($P_3 \sim iK_3, P_1 \sim iK_1, P_2 \sim iK_2$), where $\{f_1, f_2\} \equiv \partial_\varphi f_1 \, \partial_p f_2 - \partial_p f_1 \, \partial_\varphi f_2$ for any two smooth functions $f_i(\varphi, p),\ i = 1,2$. As any function $f(\varphi, p)$ periodic in φ with period 2π can – under certain conditions – be expanded in a Fourier series and as $\cos(n\varphi)$ and $\sin(n\varphi)$ can be expressed by polynomials of order n in $\cos\varphi$ and $\sin\varphi$, the observables (2) are indeed the basic ones on the phase manifold (1).

Their relationship to corresponding observables in interferences (optical or otherwise) is the following: Consider the sum

$$A = a_1 \, e^{i\,\varphi_1} + a_2 \, e^{i\,\varphi_2} \tag{5}$$

of two complex numbers, where the phases φ_i can be chosen such that $a_i > 0,\ i = 1,2$. The quantities a_i and φ_i may be functions of other parameters, e.g. space or/and time variables etc. The absolute square of A has the form

$$|A|^2(a_1, a_2, \varphi = \varphi_2 - \varphi_1) = (a_1)^2 + (a_2)^2 + 2\, a_1 \, a_2 \, \cos\varphi \ . \tag{6}$$

The corresponding "quadrature" quantity is

$$|A|^2(a_1, a_2, \varphi + \frac{\pi}{2}) = (a_1)^2 + (a_2)^2 - 2\, a_1 \, a_2 \, \sin\varphi \ , \tag{7}$$

obtained by an appropriate $\pi/2$-phase shift of either φ_1 or φ_2.

Knowing the quantities $p = a_1 \, a_2 > 0,\ a_1 \, a_2 \, \cos\varphi$ and $-a_1 \, a_2 \, \sin\varphi$ allows for a complete description of the classical interference pattern against the uniform intensity background $(a_1)^2 + (a_2)^2$. Thus, the basic observables of an interference pattern generate the Lie algebra $\mathcal{L}SO^{\uparrow}(1,2)$!

It is essential to realize that a group theoretical quantization does *not* assume that the generators of the basic Lie algebra themselves may be expressed by some conventional canonical variables like in the case of angular momentum. This may be the case locally in special examples, but in general, like in the case of the manifold (1), it will not be possible globally. For more details see the discussion below and the Refs. [1, 4].

In order to calculate expectation values and fluctuations we have to know the actions of the operators $K_i, i = 1,2,3$ on the Hilbert spaces associated with the positive discrete series of the irreducible unitary representations of $SO^{\uparrow}(1,2)$ (or its covering groups). In the following I rely heavily on Ref. [1] where more (mathematical) details and Refs. to the corresponding literature can be found.

As the eigenfunctions of K_3 – the generator of the compact subgroup – form a complete basis of the associated Hilbert spaces, it is convenient to use them as a starting point. The operators $K_+ = K_1 + iK_2$, $K_- = K_1 - iK_2$ act as ladder operators. The positive discrete series is characterized by the property that there exists a state $|k,0\rangle$ for which $K_-|k,0\rangle = 0$. The number $k > 0$ characterizes the representation: For a general normalized eigenstate $|k,n\rangle$ of K_3 we have

$$K_3|k,n\rangle = (k+n)|k,n\rangle \ , \ n = 0,1,\ldots, \tag{8}$$

$$K_+|k,n\rangle = \omega_n\,[(2k+n)(n+1)]^{1/2}|k,n+1\rangle \ , \quad |\omega_n| = 1 \ , \tag{9}$$

$$K_-|k,n\rangle = \frac{1}{\omega_{n-1}}[(2k+n-1)n]^{1/2}|k,n-1\rangle \ . \tag{10}$$

In irreducible unitary representations the operator K_- is the adjoint operator of K_+ : $(f_1, K_+f_2) = (K_-f_1, f_2)$. The phases ω_n serve to guarantee this property. Their choice depends on the concrete realization of the representations. In the examples discussed in Ref. [1] they have the values 1 or i. In the following we assume ω_n to be independent of $n : \omega_n = \omega$.

The Casimir operator $Q = K_1^2 + K_2^2 - K_3^2$ has the eigenvalues $q = k(1-k)$. The allowed values of k depend on the group: For $SO^{\uparrow}(1,2)$ itself one has $k = 1,2,\ldots$ and for the double covering $SU(1,1)$ $k = 1/2, 1, 3/2, \ldots$. The appropriate choice will depend on the physics to be described.

The relation (9) implies

$$|k,n\rangle = \omega^{-n}\left[\frac{\Gamma(2k)}{n!\,\Gamma(2k+n)}\right]^{1/2}(K_+)^n|k,0\rangle \ . \tag{11}$$

The expectation values of the self-adjoint operators $K_1 = (K_+ + K_-)/2$ and $K_2 = (K_+ - K_-)/2i$ (which correspond to the classical observables $p\cos\varphi$ and $-p\sin\varphi$) with respect to the eigenstates $|k,n\rangle$ and the associated fluctuations may be calculated with the help of the relations (8)-(10) :

$$\langle k,n|K_i|k,n\rangle = 0 \ , \quad i = 1,2 \ . \tag{12}$$

The corresponding fluctuations are

$$(\Delta K_i)^2_{k,n} = \langle k,n|K_i^2|k,n\rangle = \frac{n}{2}(2k+n) + \frac{k}{2} \ , \ i = 1,2 \ . \tag{13}$$

For very large n the correspondence principle, $(p\cos\varphi)^2 + (p\sin\varphi)^2 = p^2$, is fulfilled:

$$\langle k,n|K_1^2|k,n\rangle + \langle k,n|K_2^2|k,n\rangle \asymp n^2 \asymp \langle k,n|K_3^2|k,n\rangle \tag{14}$$

This follows already from the Casimir operator which for an irreducible representation can be rewritten as $(K_1)^2 + (K_2)^2 = (K_3)^2 + k(1-k)$. For $k = 1$ we even have $(K_1)^2 + (K_2)^2 = (K_3)^2$.

Next I define the self-adjoint operators [5] $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ as follows:

$$\widehat{\cos\varphi} = \frac{1}{2}(K_3^{-1}K_1 + K_1K_3^{-1}) \; , \quad \widehat{\sin\varphi} = -\frac{1}{2}(K_3^{-1}K_2 + K_2K_3^{-1}) \; . \tag{15}$$

Notice that K_3^{-1} is well-defined because K_3 is a positive definite operator for the positive discrete series. One has $K_3^{-1}|k,n\rangle = |k,n\rangle/(n+k)$. Using again the relations (8)-(10) we get

$$\widehat{\cos\varphi}|k,n\rangle = \frac{\omega}{4}f_{n+1}^{(k)}|k,n+1\rangle + \frac{1}{4\omega}f_n^{(k)}|k,n-1\rangle \; , \tag{16}$$

$$\widehat{\sin\varphi}|k,n\rangle = -\frac{\omega}{4i}f_{n+1}^{(k)}|k,n+1\rangle + \frac{1}{4i\omega}f_n^{(k)}|k,n-1\rangle \; , \tag{17}$$

$$f_n^{(k)} = [n(2k+n-1)]^{1/2}\left(\frac{1}{k+n} + \frac{1}{k+n-1}\right) \; , \tag{18}$$

and therefore

$$\langle k,n|\widehat{\cos\varphi}|k,n\rangle = 0 \; , \quad \langle k,n|\widehat{\sin\varphi}|k,n\rangle = 0 \tag{19}$$

and

$$\langle k,n|(\widehat{\cos\varphi})^2|k,n\rangle = \langle k,n|\,(\widehat{\sin\varphi})^2|k,n\rangle = \frac{1}{16}[(f_{n+1}^{(k)})^2 + (f_n^{(k)})^2] \tag{20}$$

$$\langle k,n|[\,\widehat{\sin\varphi},\widehat{\cos\varphi}\,]|k,n\rangle = \frac{1}{8i}[(f_{n+1}^{(k)})^2 - (f_n^{(k)})^2] \; . \tag{21}$$

For the ground state $|k,n=0\rangle$ one gets

$$\langle k,0|(\widehat{\cos\varphi})^2|k,0\rangle = \langle k,0|(\widehat{\sin\varphi})^2|k,0\rangle = \frac{(2k+1)^2}{8\,k(k+1)^2} \; . \tag{22}$$

It follows that an upper bound $\langle k,0|(\widehat{\cos\varphi})^2|k,0\rangle \leq 1$ implies for k the lower bound $k \geq k_1 \equiv [(0.5 + 0.5\sqrt{23/27}\,)^{1/3} + (0.5 - 0.5\sqrt{23/27}\,)^{1/3} - 1]/2 = 0.162\ldots$. A slightly higher lower bound for allowed values of k will be discussed below.

For very large n we have the (correct) correspondence principle limits

$$\langle k,n|(\widehat{\cos\varphi})^2|k,n\rangle = \langle k,n|(\widehat{\sin\varphi})^2|k,n\rangle \asymp \frac{1}{2}(1+O(n^{-2})) \text{ for } n \to \infty \; , \tag{23}$$

$$\langle k,n|[\,\widehat{\sin\varphi},\widehat{\cos\varphi}\,]|k,n\rangle \asymp O(n^{-2}) \quad \text{for} \quad n\to\infty\ . \tag{24}$$

I next turn to some properties of coherent states. Contrary to the conventional coherent states (i.e. the eigenstates of the Bose annihilation operator associated with the harmonic oscillator, see e.g. the reviews [6, 7] and the modern exposition [8]) there are several inequivalent ways [9] [7] to define coherent states related to the group $SO^{\uparrow}(1,2)$ or $SU(1,1)$ (see also the Refs. [11]). For our purposes the definition [9]

$$K_-|z\rangle = z\,|z\rangle\ , \quad z\in\mathbb{C}\ , \tag{25}$$

seems to be an interesting one: Using the property (11) we get

$$\langle k,n|z\rangle = \frac{1}{\bar\omega^n}\left[\frac{\Gamma(2k)}{n!\,\Gamma(2k+n)}\right]^{1/2} z^n\,\langle k,0|z\rangle\ , \tag{26}$$

($\bar\omega$: compl. conj. of ω) so that

$$\begin{aligned}\langle z|z\rangle &= \sum_{n=0}^{\infty} = \langle z|k,n\rangle\langle k,n|z\rangle = \Gamma(2k)|\langle k,0|z\rangle|^2\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\,\Gamma(2k+n)}\\ &= \Gamma(2k)|\langle k,0|z\rangle|^2|z|^{1-2k}\,I_{2k-1}(2|z|)\ ,,\end{aligned} \tag{27}$$

where

$$I_\nu(x) = \left(\frac{x}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{2n} \tag{28}$$

is the usual modified Bessel function of the first kind [10] which has the asymptotic expansion

$$I_\nu(x) \asymp \frac{e^x}{\sqrt{2\pi\,x}}\left(1-\frac{4\nu^2-1}{8x}+O(x^{-2})\right) \text{ for } x\to+\infty\ . \tag{29}$$

If $\langle z|z\rangle = 1$ we have

$$|\langle k,0|z\rangle|^2 \equiv |C_z|^2 = \frac{|z|^{2k-1}}{\Gamma(2k)\,I_{2k-1}(2|z|)}\ . \tag{30}$$

Choosing the phase of C_z appropriately and absorbing the phase ω into a redefinition of z we finally get the expansion

$$|z\rangle = \frac{|z|^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\,\Gamma(2k+n)}}\,|k,n\rangle\ . \tag{31}$$

Notice that $|z=0\rangle = |k, n=0\rangle$.
Two different coherent states are not orthogonal. They are complete, however, because with $z = \rho\, e^{i\,\alpha}$ we have the completeness relation

$$\frac{2}{\pi}\int_0^\infty d\rho\,\rho\,K_{2k-1}(2\rho)\,I_{2k-1}(2\rho)\int_0^{2\pi} d\alpha\,\langle k,n|z=\rho\,e^{i\alpha}\rangle\langle z=\rho\,e^{i\alpha}|k,n\rangle$$
$$= \langle k,n|k,n\rangle = 1\ , \tag{32}$$

where $K_\nu(x)$ is the modified Bessel function of the third kind [10].

The following expectation values are associated with the states $|z\rangle$:

$$\langle K_3\rangle_z \equiv \langle z|K_3|z\rangle = k + |z|\frac{I_{2k}(2|z|)}{I_{2k-1}(2|z|)}\ , \tag{33}$$

$$\langle K_3^2\rangle_z = k^2 + |z|^2 + |z|\frac{I_{2k}(2|z|)}{I_{2k-1}(2|z|)}\ , \tag{34}$$

so that

$$(\Delta K_3)_z^2 = |z|^2\left(1 - \frac{I_{2k}^2(2|z|)}{I_{2k-1}^2(2|z|)}\right) + (1-2k)|z|\frac{I_{2k}(2|z|)}{I_{2k-1}(2|z|)}\ . \tag{35}$$

For very large $|z|$ we have the leading terms

$$\langle K_3\rangle_z \asymp |z|\ ,\quad (\Delta K_3)_z^2 \asymp \frac{1}{2}|z| \text{ for } |z|\to+\infty\ . \tag{36}$$

This, together with the probability

$$|\langle k,n|z\rangle|^2 = \frac{|z|^{2(n+k)-1}}{n!\,\Gamma(2k+n)\,I_{2k-1}(2|z|)} \asymp 2\sqrt{\pi}\frac{|z|^{2(n+k)-1/2}}{n!\,\Gamma(2k+n)}e^{-2|z|} \tag{37}$$

shows that the corresponding distribution for large $|z|$ is not Poisson-like!

In addition we have the following expectation values

$$\langle K_1\rangle_z = \frac{1}{2}(\bar{z}+z) = \rho\,\cos\alpha\quad ,\quad \langle K_2\rangle_z = \frac{1}{2i}(\bar{z}-z) = -\rho\,\sin\alpha\ , \tag{38}$$

$$(\Delta K_1)_z^2 = (\Delta K_2)_z^2 = \frac{1}{2}\langle K_3\rangle_z\ . \tag{39}$$

Comparing Eqs. (2) and (38) we see the close relationship between the expectation values $\langle K_i\rangle_z, i=1,2$ and their classical counterparts. This supports

the above choice (25) as coherent states. Further support comes from their property to realize the minimal uncertainty relation: ¿From the third commutator in Eqs. (3) we get the general inequality

$$(\Delta K_1)^2(\Delta K_2)^2 \geq \frac{1}{4}|\langle K_3\rangle|^2 \; . \tag{40}$$

The relations (39) show that the coherent states (31) realize the minimum of the uncertainty relation (40). One can, of course, extend the discussion to associated squeezed states [11].

For the operators $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ we have (from Eqs. (16), (17) and (31)) the expectation values

$$\langle\widehat{\cos\varphi}\rangle_z = \frac{\bar{z}+z}{4|z|}\frac{g^{(k)}(|z|)}{I_{2k-1}(2|z|)} = \frac{1}{2}\cos\alpha\,\frac{g^{(k)}(|z|)}{I_{2k-1}(2|z|)} \; , \tag{41}$$

$$\langle\widehat{\sin\varphi}\rangle_z = \frac{-\bar{z}+z}{4i|z|}\frac{g^{(k)}(|z|)}{I_{2k-1}(2|z|)} = \frac{1}{2}\sin\alpha\,\frac{g^{(k)}(|z|)}{I_{2k-1}(2|z|)} \; , \tag{42}$$

$$g^{(k)}(|z|) = \sum_{n=0}^{\infty}\frac{|z|^{2(n+k)}}{n!\,\Gamma(2k+n)}\left(\frac{1}{n+k}+\frac{1}{n+k+1}\right) \; . \tag{43}$$

One has

$$g^{(k)}(|z|) = \int_0^{2|z|} du\, I_{2k-1}(u) + \frac{1}{4|z|^2}\int_0^{2|z|} du\, u^2 I_{2k-1}(u) \; . \tag{44}$$

The right hand side may be expressed by a combination of modified Bessel and Lommel functions [12]. For large $|z|$ one obtains [13]

$$\frac{g^{(k)}(|z|)}{I_{2k-1}(2|z|)} \asymp 2\left(1-\frac{1}{4|z|}+O(|z|^{-2})\right) \text{ for large } |z| \tag{45}$$

which again gives the expected correspondence principle limits for $\langle\widehat{\cos\varphi}\rangle_z$ and $\langle\widehat{\sin\varphi}\rangle_z$.

The expectation values $\langle\widehat{\cos\varphi}^2\rangle_z$ etc. may be calculated by observing that $\langle\widehat{\cos\varphi}^2\rangle_z = \sum_n \langle z|\widehat{\cos\varphi}|k,n\rangle\langle k,n|\widehat{\cos\varphi}|z\rangle = \sum_n |\langle k,n|\widehat{\cos\varphi}|z\rangle|^2$ etc.

The operators $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ are bounded self-adjoint operators (see Eqs. (20)) with a continuous spectrum within the interval $[-1,+1]$ for $k \geq 0.32$. The last assertion follows from Eqs. (41) and (42) together with a numerical analysis of the ratio $g^{(k)}(|z|)/I_{2k-1}(2|z|)$ which shows that ratio to be < 2

for all finite $|z|$ if $k \geq 0.32$. That is not so e.g. for $k = 0.25$. This and the relation (45) imply that at least for $k \geq 0.32$ we have $\sup_z |\langle\widehat{\cos\varphi}\rangle_z| = \sup_z |\langle\widehat{\sin\varphi}\rangle_z| = 1$ from which the support of the spectrum follows [14]. Thus, for the groups $SO^{\uparrow}(1,2)$ and $SU(1,1)$ which have $k = 1$ and $k = 1/2$ respectively as their lowest k-values we are on the safe side.

The ansatz $|\mu\rangle = \sum_{n=0}^{\infty} a_n\,|k,n\rangle$ for the improper "eigenfunctions" of $\widehat{\cos\varphi}$ with "eigenvalues" μ, $\widehat{\cos\varphi}|\mu\rangle = \mu\,|\mu\rangle$, leads to the recursion formula $a_{n+1} = (4\,\mu\,a_n - f_n^{(k)}\,a_{n-1})/f_{n+1}^{(k)}\,,\ f_0^{(k)} = 0\,,$ which allows to express the a_n by μ, a_0 and the $f_n^{(k)}$.

Up to now I have not specified the concrete form of the Hilbert space, the operators $K_i, i = 1,2,3$ and the eigenfunctions $|k,n\rangle$. Several interesting examples may be found in Ref. [1].

I finally come – very preliminary – to some subtle points of the physical interpretation of the results. Let us take the example represented by the Eqs. (6) and (7): Here the eigenvalues of the operator K_3 correspond to the square root $\sqrt{I_1\,I_2}$ of the product of the intensities I_1 and I_2 of the two interfering classical waves. In an interference experiment with photons we therefore expect the natural number n characterizing the eigenvalues $n+k$ of K_3 to count the number of photons registered by an appropriate device. The number k characterizes the ground state and it is not so obvious which value it will take. The group theoretical quantization requires its own interpretation.

On the other hand we are very much used to the conventional quantization in which the intensities I_1 and I_2 become proportional to associated number operators $\hat{N}_1$ and $\hat{N}_2$ built up in the simplest case from two pairs of bosonic creation and annihilation operators b_i^+ and b_i, $i = 1,2$ which may also be used to construct the associated Hilbert space: $\hat{N}_i^{(b)} = b_i^+ b_i$ and $b_i^+|n_i\rangle_i = \sqrt{n_i+1}\,|n_i+1\rangle_i\,, b_i|n_i\rangle_i = \sqrt{n_i}\,|n_i-1\rangle_i$. One can take the square root of $\hat{N}_i^{(b)}$ naively or (in order to have those square roots linear in the b_i^+ and b_i) in a Dirac-type manner:

$$\sqrt{\hat{N}_i^{(b)}} = \sigma_+\,b_i^+ + \sigma_- b_i = \begin{pmatrix} 0 & b_i^+ \\ b_i & 0 \end{pmatrix}\ ,\quad \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)\ , \tag{46}$$

$$\sqrt{\hat{N}_i^{(b)}}|\sqrt{n_i}\,\rangle_i = \sqrt{n_i}\,|\sqrt{n_i}\,\rangle_i\ ,\ |\sqrt{n_i}\,\rangle_i = \begin{pmatrix} |n_i\rangle_i \\ |n_i-1\rangle_i \end{pmatrix}\ , \tag{47}$$

$$\begin{pmatrix} 0 & b_i^+ \\ b_i & 0 \end{pmatrix}^2 = \begin{pmatrix} \hat{N}_i^{(b)} & 0 \\ 0 & \hat{N}_i^{(b)}+1 \end{pmatrix}\ ,\ i = 1,2\ ,$$

where $\sigma_i, i = 1,2$ are the usual Pauli matrices. If we now consider the operator $\sqrt{\hat{N}_1^{(b)}} \otimes \sqrt{\hat{N}_2^{(b)}}$ acting on $|\sqrt{n_1}\,\rangle_1 \otimes |\sqrt{n_2}\,\rangle_2$ to be the quantized counterpart of $\sqrt{I_1\, I_2}$ then we obviously can have the eigenvalues $n \in \mathbb{N}_0$ for all n_1, n_2 only if $n_1 = n_2$, that is for an ideal 50% beam splitter. Another possibility is that $n_2 \neq n_1$ but $\sqrt{n_1\, n_2} = n = 0,1,2,\ldots$.

Now there is a close relationship between another pair $a_i^+, a_i, i = 1,2$, of bosonic creation and annihilation operators and the irreducible unitary representations of $SU(1,1)$ (see Ref. [1] and the literature quoted there): The operators

$$K_3^{(a)} = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1)\ , \quad K_+^{(a)} = a_1^+ a_2^+\ , \quad K_-^{(a)} = a_1 a_2 \tag{48}$$

obey the associated commutation relations (3) and the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ of the two harmonic oscillator Hilbert spaces contains all the irreducible unitary representations of the group $SU(1,1)$ in the following way: Let $|m_i\rangle_i,\ m_i \in \mathbb{N}_0, i = 1,2$, be the eigenstates of the number operators $\hat{N}_i^{(a)}$ generated by a_i^+ from the oscillator ground states. Then each of those two subspaces of $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc} = \{|m_1\rangle_1 \otimes |m_2\rangle_2\}$ for which $|m_1 - m_2| \neq 0$ is fixed contains an irreducible representation with $k = 1/2 + |m_1 - m_2|/2 = 1, 3/2, 2, \ldots$ and for which the number n in the eigenvalue $n + k$ is given by $\min\{m_1, m_2\}$. For the "diagonal" case $m_2 = m_1$ one gets the unitary representation with $k = 1/2$. In general the two systems of oscillators a_i^+, a_i and b_i^+, b_i will represent different physical systems, but perhaps there can be a correspondence between the diagonal states $|\sqrt{n_1}\,\rangle_1 \otimes |\sqrt{n_2 = n_1}\,\rangle_2$ and $|m_1\rangle_1 \otimes |m_2 = m_1\rangle_2$.

This suggests that the representation with $k = 1/2$ may play a special role for the applications. Such a suggestion is supported intuitively by the following argument: For $k = 1/2$ the operator K_3 has the same spectrum $\{n + 1/2,\ n \in \mathbb{N}_0\}$ as the harmonic oscillator. Since n counts the number of quanta (e.g. photons) the additional term $1/2$ will represent the usual ground state contribution to the energy! Experiments will have to decide which value of k one has to choose.

Finally I would like to stress again that a group theoretical quantization like the one above does *not* suppose that there is a "deeper" conventional canonical structure in terms of the usual q and p. It claims to provide an appropriate quantization for topologically nontrivial symplectic manifolds like (1) by itself. Quantum optics (or other quantum interference phenomena) may very well be able to test such claims experimentally. In addition it may

test the identification (15) as an operator version of $\cos\varphi$ and $\sin\varphi$. That definition is a new ansatz *within* - not a basic ingredient *of* - the group theoretical quantization scheme.

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